Kummer Rigidity for Hyperbolic Hyperkähler Automorphisms

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Outline

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The Cat Map

Let $\mathbb{T} = \mathbb{R}^2/\mathbb{Z}^2$.

$$\begin{aligned} Cat \colon \mathbb{T} &\to \mathbb{T}, \\ (z, w) &\mapsto (2z + w, z + w), \qquad \text{i.e.,} \\ \begin{bmatrix} z \\ w \end{bmatrix} &\mapsto \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} z \\ w \end{bmatrix} \end{aligned}$$

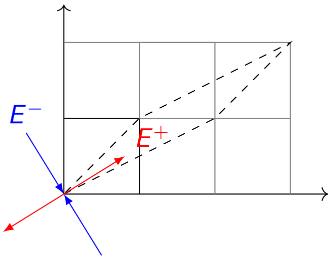
This *Cat* will be a toy example for a while.

Cat is a matrix $\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$. Two eigenlines: $(\varphi = \frac{1}{2}(1+\sqrt{5}))$.

Example (Stable and Unstable distributions)

φ^{-2}-eigenline $E^-(x) \subset T_x\mathbb{T}$: shrinks by *Cat*. *Stable distribution* of *Cat*.

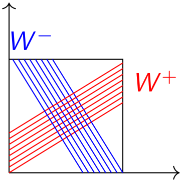
φ^{2}-eigenline $E^+(x) \subset T_x\mathbb{T}$: expands by *Cat*. *Unstable distribution* of *Cat*.



Example (Stable and Unstable manifolds)

Wrap $E^-(x), E^+(x) \subset T_x\mathbb{T}$ on $\mathbb{T}$. Then we get immersed manifolds $W^-(x), W^+(x)$ (which are $\cong \mathbb{R}$).

Called *stable* and *unstable manifold* of *Cat*.



Lyapunov exponents encodes how *Cat* acts on $E^{\pm}(x)$ (and $W^{\pm}(x)$).

Example (Lyapunov Exponents)

$\log \varphi^2 = .9624$ and $\log \varphi^{-2} = -.9624$ are *Lyapunov exponents* of *Cat*.

They are log of dialation rates along E^+ and E^-:

- *Cat* stretches $v \in E^+(x)$ by φ^2.
- *Cat* shrinks $v \in E^-(x)$ by φ^{-2}.

Entropy

X compact, $f : X \to X$ continuous, μ a f-invariant measure.

- *Entropy* $h(f)$ measures how complicated orbits of f are.
- *Measure entropy* $h_\mu(f)$ measures 'information' of orbits w.r.t. μ.

Example

If $f : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ shifts decimal point, $f(.a_1 a_2 \dots) = .a_2 a_3 \dots$, then

$$(\text{patterns of } x, f(x), \dots, f^{N-1}(x)) \sim (\text{first } N \text{ digits } x = .a_1 a_2 \dots a_N \cdots).$$

Thus 10^N many, and $h(f) = \log 10$.

If $\mu = \text{Leb}$, the patterns are uniform, so $h_\mu(f) = \log 10$ too.

μ is a *measure of maximal entropy* (m.m.e.) if $h_\mu(f) = h(f)$.

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For Cat, we use Lyapunov exponents to get $h(Cat)$.

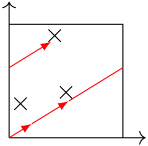
Example (Entropy)

Entropy $h(Cat)$ of Cat is $\log \varphi^2$.

Orbits $Cat^N(x)$ lie near $W^+(x)$, so

$$(\text{how long } W^+(x) \text{ is after } Cat^N) \sim (e^{\log \varphi^2})^N = (\varphi^2)^N$$

is the complexity.



Recall μ is a *measure of maximal entropy* (m.m.e.) if $h_\mu(f) = h(f)$.

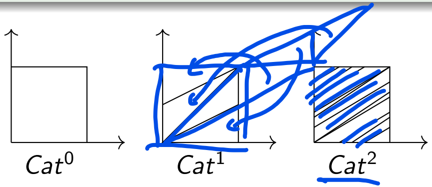
Example (m.m.e.)

For Cat, the volume on $\mathbb{T}$ is a m.m.e..

$h_{\text{vol}}(Cat)$ counts how many 'slices' does Cat^N make on $[0,1]^2$: $\sim (\varphi^2)^N$, thus

$$h_{\text{vol}}(Cat) = \log \varphi^2 = h(Cat),$$

so vol is m.m.e.



Summary

Cat map features:

- Stable/Unstable distributions $E^{\pm}$ and manifolds $W^{\pm}$.
- *Cat* acts dilations along $E^{\pm}$,
- and log of dilation rates are Lyapunov exponents.
- An m.m.e. of *Cat* is the volume measure.

Same properties, if we

- complexify $Cat_{\mathbb{C}}: \mathbb{C}^2/\mathbb{Z}^4 \to \mathbb{C}^2/\mathbb{Z}^4$, and
- n-fold product $Cat_{\mathbb{C}}^{\times n}: \mathbb{C}^{2n}/(\mathbb{Z}^4)^n \to \mathbb{C}^{2n}/(\mathbb{Z}^4)^n$.

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Kummer Example

Let $\mathbb{T}_\mathbb{C} = \mathbb{C}^2/\mathbb{Z}^4$, complexified $Cat_\mathbb{C} \colon \mathbb{T}_\mathbb{C} \to \mathbb{T}_\mathbb{C}$.

Let $\mathbb{T}_\mathbb{C}/\{\pm 1\} = \mathbb{T}_\mathbb{C}/(x \sim -x)$. Induce $Cat_\mathbb{C} \colon \mathbb{T}_\mathbb{C}/\{\pm 1\} \to \mathbb{T}_\mathbb{C}/\{\pm 1\}$.

- $\mathbb{T}_\mathbb{C}/\{\pm 1\}$ is like a complex manifold, but with singularities.
- *Normalization* $\pi \colon X \to \mathbb{T}_\mathbb{C}/\{\pm 1\}$ is a surjective map from a complex manifold X, with a dense (Zariski) open $U \subset X$ that
  - image of U by π is the non-singular locus of $\mathbb{T}_\mathbb{C}/\{\pm 1\}$, and
  - π is an isomorphism $U \cong \pi(U)$.

  And, any holomorphic $g \colon Y \to \mathbb{T}_\mathbb{C}/\{\pm 1\}$ factors $Y \to X \to \mathbb{T}_\mathbb{C}/\{\pm 1\}$.

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Normalize $X \to \mathbb{T}_{\mathbb{C}}/\{\pm 1\}$. Then $Cat_{\mathbb{C}}$ lifts to $f\colon X \to X$.

(In short: (X, f) normalizes and lifts $(\mathbb{T}_{\mathbb{C}}/\{\pm 1\}, Cat_{\mathbb{C}})$.)

- X—*Kummer surface* from $\mathbb{T}_{\mathbb{C}}$.
- f—*Kummer example* from $Cat_{\mathbb{C}}\colon \mathbb{T}_{\mathbb{C}} \to \mathbb{T}_{\mathbb{C}}$.

Features of *Cat* are mostly kept on Kummer examples.

Proposition

The Kummer example $f : X \to X$ from $Cat_{\mathbb{C}}$ has:

- *Stable/Unstable distributions $E^{\pm}$ and manifolds $W^{\pm}$, defined on a dense (Zariski) open $U \subset X$.*
- *f acts dilations along $E^{\pm}$: a metric ω_0 on U has $f^*\omega_0|E^{\pm} = \lambda_{\pm} \cdot \omega_0|E^{\pm}$,*
- *and $\frac{1}{2}$ of log of dilation rates are Lyapunov exponents.*
- *An m.m.e. of f is the volume measure on X.*

Outline

Object of Interest

Now focus on a pair (X, f):

- (X, ω) compact Kähler manifold,
  - complex manifold X with a closed, nondegenerate real 2-form ω
- $f : X \to X$ holomorphic automorphism.

Cat-like system

Definition

Call (X, f) *Cat-like* if there is a dense Zariski open $U \subset X$ with,

- Stable/Unstable distributions $E^{\pm}$ and manifolds $W^{\pm}$, on U.
- A Ricci-flat Kähler metric ω_0 on U, with $f^*\omega_0|E^{\pm} = \lambda_{\pm} \cdot \omega_0|E^{\pm}$.
- $\frac{1}{2}\log\lambda_+, \frac{1}{2}\log\lambda_-$ are the only Lyapunov exponents.
- An m.m.e. of f is the volume measure $\omega_0^{\dim X}$.

Example (Seen by far)

(1) Complexified $Cat_{\mathbb{C}}$ map, (2) Kummer examples from $Cat_{\mathbb{C}}$.

Questions

1. Any other *Cat*-like systems, other than those from $Cat_{\mathbb{C}}$ as above?
2. When a system is *Cat*-like?

Some Answers

Theorem (J. '22)

1. *If a Cat-like system (X, f) is based on a projective hyperkähler manifold X, then it is a "Kummer example."*
2. *If X is hyperkähler, f has $h(f) > 0$, and if a m.m.e. $\mu \ll \mathrm{vol} = \omega^{\dim X}$, then (X, f) is Cat-like.*

New terms, new hypotheses. . . where they are from?

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Contexts

Think of a dynamical system on a manifold (M, f), whose measure of maximal entropy (m.m.e.) μ is in volume class.

- Typically such a system has locally homogeneous structures.
- If M is a complex manifold, usually (M, f) comes from a torus. (e.g. Kummer example (X, f)).

$$(X, f) \longrightarrow (\mathbb{T}_{\mathbb{C}}/\{\pm 1\}, Cat_{\mathbb{C}}) \longleftarrow (\mathbb{T}_{\mathbb{C}}, Cat_{\mathbb{C}})$$

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Some results with the philosophy:

- (Zdunik '90) If $f\colon \mathbb{P}^1 \to \mathbb{P}^1$ has m.m.e. $\mu \ll$ Leb, then f is Lattès.
- (Berteloot–Dupont '05) Same result, but $f\colon \mathbb{P}^k \to \mathbb{P}^k$.
- (Cantat–Dupont '20) If $f\colon X \to X$, where X is a projective surface, $h(f) > 0$, has m.m.e. $\mu \ll$ vol, then f is a Kummer example.
- (Filip–Tosatti '21) Same result, but X any K3 surface.
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Hyperkähler Manifolds

Definition

A simply connected compact Kähler manifold (X, ω) is *hyperkähler* if the group $H^0(X, \Omega^2)$ is generated by a 'holomorphic symplectic form' Ω.

- X then has even dimension, $2n = \dim X$.
- ω above is *Ricci-flat*. Equivalently, $\omega^{2n} = (\Omega\overline{\Omega})^n =: \text{vol}$.

Example

K3 surfaces ($2n = 2$). Hilbert scheme of n points on a K3 surface.

"Kummer example"

The term "Kummer example" can be generalized to higher dimensions.

Definition

(X, f) is a *Kummer example* if

- we have a torus $\mathbb{T} = \mathbb{C}^m/\Lambda$ ($m = \dim X$) and an affine-linear map $L\colon \mathbb{T} \to \mathbb{T}$,
- a quotient by a finite group, $\mathbb{T}/\Gamma$, and L commutes with Γ;
- and (X, f) normalizes and lifts $(\mathbb{T}/\Gamma, L)$.

Recall $\mathbb{T}_{\mathbb{C}} = \mathbb{C}^2/\mathbb{Z}^4$.

Example

Let $L = Cat_{\mathbb{C}}^{\times n} \colon \mathbb{T}_{\mathbb{C}}^n \to \mathbb{T}_{\mathbb{C}}^n$.

Let $S_{n+1} =$ (symmetric group of $(n+1)$ letters) $\curvearrowright (\mathbb{C}^2/\mathbb{Z}^4)^{n+1}$.

This S_{n+1}-action can be induced to $\mathbb{T}_{\mathbb{C}}^n$, commuting with L.

Let (X, f) normalize and lift $(\mathbb{T}_{\mathbb{C}}^n/S_{n+1}, L)$. So a Kummer example.

In fact, (X, f) is *Cat*-like (as $L = Cat_{\mathbb{C}}^{\times n}$ is).

Outline

From *Cat*-like to Kummer

Recall: X is a projective hyperkähler manifold.

- (Projective) $\Rightarrow$ Construct a contraction $X \to Y$ to a normal variety Y, for the (klt) pair $(X, X \setminus U)$.
- (Hyperkähler) $\Rightarrow$ U is flat under ω_0.
  (Flatness result in (Benoist–Foulon–Labourie '92) applies here.)

Ω ← $\mathbb{R}$-symplectic ok?

Then Y_{reg} is flat, so $Y = \mathbb{T}/\Gamma$. So (X, f) normalizes and lifts $(\mathbb{T}/\Gamma, L)$.

Showing *Cat*-like

Recall:

- X is a hyperkähler manifold,
- $f\colon X \to X$ holomorphic automorphism, with $h(f) > 0$, and
- μ is an m.m.e. of (X, f), vol-class: $\mu \ll \mathrm{vol}$.

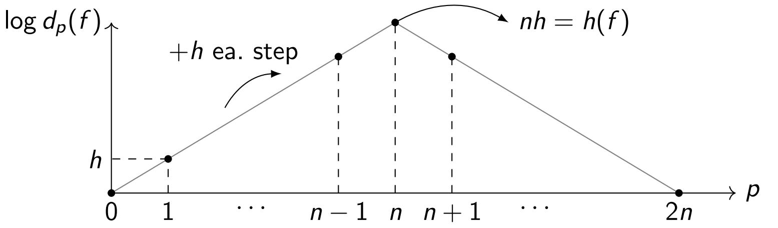
Goal: (X, f) is *Cat*-like.

Base ingredient:

Theorem (Oguiso '09)

Let X be hyperkähler of $\dim X = 2n$ and $f : X \to X$ be holomorphic automorphism. Let $d_p(f)$ be the spectral radius of $f^* \circlearrowright H^{p,p}(X, \mathbb{C})$. Let $h := \log d_1(f)$.

Then for $0 \leq k \leq n$, $\log d_{2n-k}(f) = \log d_k(f) = k \cdot h$; and $h(f) = nh$.



$\pm\frac{1}{2}h$ are only Lyapunov exponents:

- Lyapunov exponents: $\chi_1 \geq \ldots \geq \chi_n \geq 0 \geq \chi_{n+1} \geq \ldots \geq \chi_{2n}$.
- (Spectral radii d_p's) + (Ledrappier–Young formula) + $\mu \ll \mathrm{vol} \Rightarrow$

$$h(f) = 2(\chi_1 + \ldots + \chi_n) \geq 2n\chi_n \geq nh = h(f).$$

- Thus $\chi_1 = \ldots = \chi_n = \frac{1}{2}h$, and sim. $\chi_{n+1} = \ldots = \chi_{2n} = -\frac{1}{2}h$.

Stable/Unstable distributions $E^{\pm}$ and manifolds $W^{\pm}$:

- Defined μ-a.e. points.

Measure of maximal entropy μ is vol:

- Cocycle $(N, x) \mapsto e^{Nh/2} D_x f^N | E^-(x)$ is, up to a bounded conjugation, *unitary* valued. Sim. for $Df^{-N}|E^+$.
- $\therefore f$ uniformly hyperbolic
  $\Rightarrow W^{\pm}$'s widen the support S of $\mu = \frac{1}{|S|}\mathrm{vol}|_S$
  $\Rightarrow \mu = \mathrm{vol}$.

A metric with $f^*\omega_0|E^\pm = e^{\pm h}\omega_0|E^\pm$:

- ω_0 is the limit of metrics

$$\omega_k = \frac{1}{2k+1}\sum_{i=-k}^{k} e^{-|i|h}(f^i)^*\omega + \sqrt{-1}\partial\bar{\partial}\phi_k,$$

  - i.e., ω_0 is like the Lyapunov metric;
  - ϕ_k's are set so that ω_k's are Ricci-flat ($\omega_k^{2n} = \mathrm{vol}$).
  - Convergence holds at least on a Zariski open set U.
- (Jensen's inequality) + (Cohomology calculus) on ω_k's, gives $f^*\omega_0|E^\pm = e^{\pm h}\omega_0|E^\pm$.